A COUNTEREXAMPLE FOR A SUP THEOREM IN NORMED SPACES

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ABSTRACT

An example is given of a normed linear space that is not complete but for which each continuous linear functional attains its supremum on the unit ball.

Since the unit ball of a Banach space is weakly compact if and only if the space is reflexive, it follows that for a reflexive space each continuous linear functional attains its sup on the unit ball. It also is true that a Banach space is reflexive if each continuous linear functional attains its sup on the unit bal [2, p. 215]. It is natural to ask whether there exists an incomplete normed linear space for which each continuous linear functional attains its sup on the unit ball. The completion of such a space must be reflexive. Moreover, it is clear that the completion cannot have the property that each continuous linear functional attains its sup at exactly one point of the unit ball—thus the completion cannot be Hilbert space or any space whose unit ball is strictly convex.

It follows from the Krein-Milman theorem that the linear span X of the extreme points of the unit ball of a reflexive Banach space B is dense in B and that each continuous linear functional on B attains its sup on the unit ball at points of a closed convex set that contains points of X. Thus if $X \neq B$, then X is an incomplete space for which each continuous linear functional attains its sup on the unit ball.

The following example was constructed several years ago in response to an inquiry by Frank Deutsch, who was interested in whether a normed linear space is complete if for each closed convex subset K and each point x in the space there

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is a point x_0 in K closest to x. It would have followed that this conjecture is true if it had been true that each incomplete space has a continuous linear functional that does not attain its sup on the unit ball.

Example of an incomplete normed linear space for which each continuous linear functional attains its sup on the unit ball. Let B be a countable Hilbert product of increasing dimensional c_0-spaces, so that the members of B are of type $x = (x_1^1; x_1^2, x_2^2; x_1^3, x_2^3, x_3^3; \cdots)$, with

$$\| x \| = [|x_1^1|^2 + (\sup\{|x_1^2|, |x_2^2|\})^2 + (\sup\{|x_1^3|, |x_2^3|, |x_3^3|\})^2 + \cdots]^{\frac{1}{2}}. \tag{1}$$

Let X be the linear span of all members x of B for which

$$|x_1^n| = |x_2^n| = \cdots = |x_n^n| \text{ for all } n. \tag{2}$$

Since B is a Hilbert product of reflexive spaces, B is reflexive [1, p. 31]. Therefore the Krein-Milman theorem and the fact that X is the linear span of the set of extreme points of the unit ball of B imply that X is dense in B. Alternatively, one can note that each member of B with exactly one nonzero component is the sum of two members that satisfy (2), so that X contains all members of B with only a finite number of nonzero components and X is dense in B. If $x \in X$ and x is a linear combination of n members of X that satisfy (2), then for each $m > 2^n$ at least two of $\{x_1^m, \cdots, x_m^m\}$ are equal. Thus the sequence $(1/n)$ belongs to B but not to X, so that $X \neq B$ and X is not complete. Let f be an arbitrary continuous linear functional on B and x be such that $\|x\| = 1$ and $f(x) = \|f\|$. Then there is a sequence of numbers (f_i^n) such that

$$f(x) = f_1^1 x_1^1 + (f_1^2 x_1^2 + f_2^2 x_2^2) + (f_1^3 x_1^3 + f_2^3 x_2^3 + f_3^3 x_3^3) + \cdots. \tag{3}$$

The norm of x as given by (1) is not changed if for each n we replace each x_i^n by $\pm \sup_i |x_i^n|$, where the "+" is used if $f_i^n \geqq 0$ and the "−" if $f_i^n < 0$. The changes do not decrease the sum in (3), so the sum does not change and the "new x" is a member of the unit ball of X at which f attains its sup.

References

1. M. M. Day, *Normed Linear Spaces*, Academic Press, 1962.
2. R. C. James, *Characterizations of reflexivity*, Studia Math. 23 (1964), 205–216.

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